Fractional Exact Solutions and Solitons in Gravity

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Abstract

We survay our recent results on fractional gravity theory. It is also provided the Main Theorem on encoding of geometric data (metrics and connections in gravity and geometric mechanics) into solitonic hierarchies. Our approach is based on Caputo fractional derivative and nonlinear connection formalism.

Keywords: fractional calculus, fractional geometry, fractional gravity, nonlinear connection, nonholonomic manifold.

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Contents

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1 Introduction

Recently, we extended the fractional calculus to Ricci flow theory, gravity and geometric mechanics, solitonic hierarchies etc [1, 2, 3, 4, 5, 6]. In this work, we outline some basic geometric constructions related to fractional derivatives and integrals and their applications in modern physics and mechanics.

Our approach is also connected to a method when nonholonomic deformations of geometric structures[1] induce a canonical connection, adapted to a necessary type nonlinear connection structure, for which the matrix coefficients of curvature are constant [7, 8]. For such an auxiliary connection, it is possible to define a bi–Hamiltonian structure and derive the corresponding solitonic hierarchy.

The paper is organized as follows: In section 2, we outline the geometry of N–adapted fractional manifolds and provide an introduction to fractional gravity. In section 3, we show how fractional gravitational field equations can be solved in a general form. Section 4 is devoted to the Main Theorem on fractional solitonic hierarchies corresponding to metrics and connections in fractional gravity. The Appendix contains necessary definitions and formulas on Caputo fractional derivatives.

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[1]determined by a fundamental Lagrange/ Finsler / Hamilton generating function (or, for instance, and Einstein metric)

2 Fractional Nonholonomic Manifolds and Gravity

Let us consider a "prime" nonholonomic manifold $\mathbf{V}$ is of integer dimension $\dim \mathbf{V} = n + m, n \geq 2, m \geq 1$.[2] Its fractional extension $\overset{\alpha}{\mathbf{V}}$ is modelled by a quadruple $(\mathbf{V}, \overset{\alpha}{\mathbf{N}}, \overset{\alpha}{\mathbf{d}}, \overset{\alpha}{\mathbf{I}})$, where $\overset{\alpha}{\mathbf{N}}$ is a nonholonomic distribution stating a nonlinear connection (N–connection) structure (for details, see Appendix A with explanations for formula (1). The fractional differential structure $\overset{\alpha}{\mathbf{d}}$ is determined by Caputo fractional derivative (A.1) following formulas (A.3) and (A.4). The non–integer integral structure $\overset{\alpha}{\mathbf{I}}$ is defined by rules of type (A.2).

A nonlinear connection (N–connection) $\overset{\alpha}{\mathbf{N}}$ for a fractional space $\overset{\alpha}{\mathbf{V}}$ is defined by a nonholonomic distribution (Whitney sum) with conventional h– and v–subspaces, $\underline{h}\overset{\alpha}{\mathbf{V}}$ and $\underline{v}\overset{\alpha}{\mathbf{V}}$,

$$\overset{\alpha}{\underline{T}}\overset{\alpha}{\mathbf{V}} = \underline{h}\overset{\alpha}{\mathbf{V}} \oplus \underline{v}\overset{\alpha}{\mathbf{V}}. \tag{1}$$

A fractional N–connection is defined by its local coefficients $\overset{\alpha}{\mathbf{N}} = \{ {}^{\alpha}N_i^a \}$, when

$$\overset{\alpha}{\mathbf{N}} = {}^{\alpha}N_i^a(u)(dx^i)^{\alpha} \otimes \overset{\alpha}{\underline{\partial}}_a .$$

For a N–connection $\overset{\alpha}{\mathbf{N}}$, we can always construct a class of fractional (co) frames (N–adapted) linearly depending on ${}^{\alpha}N_i^a$,

$$\begin{aligned} {}^{\alpha}\mathbf{e}_{\beta} &= \left[{}^{\alpha}\mathbf{e}_j = \overset{\alpha}{\underline{\partial}}_j - {}^{\alpha}N_j^a \overset{\alpha}{\underline{\partial}}_a, \ {}^{\alpha}e_b = \overset{\alpha}{\underline{\partial}}_b \right], & (2) \\ {}^{\alpha}\mathbf{e}^{\beta} &= [\, {}^{\alpha}e^j = (dx^j)^{\alpha}, \ {}^{\alpha}\mathbf{e}^b = (dy^b)^{\alpha} + {}^{\alpha}N_k^b (dx^k)^{\alpha}]. & (3) \end{aligned}$$

The nontrivial nonholonomy coefficients are computed ${}^{\alpha}W_{ib}^a = \overset{\alpha}{\underline{\partial}}_b \, {}^{\alpha}N_i^a$ and ${}^{\alpha}W_{ij}^a = {}^{\alpha}\Omega_{ji}^a = {}^{\alpha}\mathbf{e}_i \, {}^{\alpha}N_j^a - {}^{\alpha}\mathbf{e}_j \, {}^{\alpha}N_i^a$ for

$$[\, {}^{\alpha}\mathbf{e}_{\alpha}, \ {}^{\alpha}\mathbf{e}_{\beta}] = {}^{\alpha}\mathbf{e}_{\alpha} \, {}^{\alpha}\mathbf{e}_{\beta} - {}^{\alpha}\mathbf{e}_{\beta} \, {}^{\alpha}\mathbf{e}_{\alpha} = {}^{\alpha}W_{\alpha\beta}^{\gamma} \, {}^{\alpha}\mathbf{e}_{\gamma}.$$

In above formulas, the values ${}^{\alpha}\Omega_{ji}^a$ are called the coefficients of N–connection curvature. A nonholonomic manifold defined by a structure $\overset{\alpha}{\mathbf{N}}$ is called, in brief, a N–anholonomic fractional manifold.

[2] A nonholonomic manifold is a manifold endowed with a non–integrable (equivalently, nonholonomic, or anholonomic) distribution.

We write a metric structure $\overset{\alpha}{\mathbf{g}} = \{ {}^{\alpha}g_{\underline{\alpha\beta}}\}$ on $\overset{\alpha}{\mathbf{V}}$ in the form

$$\begin{aligned} \overset{\alpha}{\mathbf{g}} &= {}^{\alpha}g_{kj}(x,y)\ {}^{\alpha}e^{k} \otimes {}^{\alpha}e^{j} + {}^{\alpha}g_{cb}(x,y)\ {}^{\alpha}\mathbf{e}^{c} \otimes {}^{\alpha}\mathbf{e}^{b} \\ &= \eta_{k'j'}\ {}^{\alpha}e^{k'} \otimes {}^{\alpha}e^{j'} + \eta_{c'b'}\ {}^{\alpha}\mathbf{e}^{c'} \otimes {}^{\alpha}\mathbf{e}^{b'}, \end{aligned} \tag{4}$$

where matrices $\eta_{k'j'} = diag[\pm 1, \pm 1, ..., \pm 1]$ and $\eta_{a'b'} = diag[\pm 1, \pm 1, ..., \pm 1]$, for the signature of a "prime" spacetime $\mathbf{V}$, are obtained by frame transforms $\eta_{k'j'} = e^{k}{}_{k'}\ e^{j}{}_{j'}\ {}^{\alpha}g_{kj}$ and $\eta_{a'b'} = e^{a}{}_{a'}\ e^{b}{}_{b'}\ {}^{\alpha}g_{ab}$.

A distinguished connection (d–connection) $\overset{\alpha}{\mathbf{D}}$ on $\overset{\alpha}{\mathbf{V}}$ is defined as a linear connection preserving under parallel transports the Whitney sum (1). We can associate a N–adapted differential 1–form

$$ {}^{\alpha}\mathbf{\Gamma}^{\tau}{}_{\beta} = {}^{\alpha}\mathbf{\Gamma}^{\tau}{}_{\beta\gamma}\ {}^{\alpha}\mathbf{e}^{\gamma}, \tag{5}$$

parametrizing the coefficients (with respect to (3) and (2)) in the form ${}^{\alpha}\mathbf{\Gamma}^{\gamma}{}_{\tau\beta} = \left({}^{\alpha}L^{i}_{jk},\ {}^{\alpha}L^{a}_{bk},\ {}^{\alpha}C^{i}_{jc},\ {}^{\alpha}C^{a}_{bc}\right)$.

The absolute fractional differential ${}^{\alpha}\mathbf{d} = {}_{1x}\overset{\alpha}{d}_{x} + {}_{1y}\overset{\alpha}{d}_{y}$ acts on fractional differential forms in N–adapted form; the value ${}^{\alpha}\mathbf{d} := {}^{\alpha}\mathbf{e}^{\beta}\ {}^{\alpha}\mathbf{e}_{\beta}$ splits into exterior h- and v–derivatives when

$$ {}_{1x}\overset{\alpha}{d}_{x} := (dx^{i})^{\alpha} \quad {}_{1x}\overset{\alpha}{\underline{\partial}}_{i} = {}^{\alpha}e^{j}\ {}^{\alpha}\mathbf{e}_{j} \text{ and } {}_{1y}\overset{\alpha}{d}_{y} := (dy^{a})^{\alpha} \quad {}_{1x}\overset{\alpha}{\underline{\partial}}_{a} = {}^{\alpha}\mathbf{e}^{b}\ {}^{\alpha}e_{b}.$$

The torsion and curvature of a fractional d–connection $\overset{\alpha}{\mathbf{D}} = \{ {}^{\alpha}\mathbf{\Gamma}^{\tau}{}_{\beta\gamma}\}$ can be defined and computed, respectively, as fractional 2–forms,

$$\begin{aligned} {}^{\alpha}\mathcal{T}^{\tau} &\doteqdot \overset{\alpha}{\mathbf{D}}\ {}^{\alpha}\mathbf{e}^{\tau} = {}^{\alpha}\mathbf{d}\ {}^{\alpha}\mathbf{e}^{\tau} + {}^{\alpha}\mathbf{\Gamma}^{\tau}{}_{\beta} \wedge {}^{\alpha}\mathbf{e}^{\beta} \text{ and} \\ {}^{\alpha}\mathcal{R}^{\tau}{}_{\beta} &\doteqdot \overset{\alpha}{\mathbf{D}}\ {}^{\alpha}\mathbf{\Gamma}^{\tau}{}_{\beta} = {}^{\alpha}\mathbf{d}\ {}^{\alpha}\mathbf{\Gamma}^{\tau}{}_{\beta} - {}^{\alpha}\mathbf{\Gamma}^{\gamma}{}_{\beta} \wedge {}^{\alpha}\mathbf{\Gamma}^{\tau}{}_{\gamma} = {}^{\alpha}\mathbf{R}^{\tau}{}_{\beta\gamma\delta}\ {}^{\alpha}\mathbf{e}^{\gamma} \wedge {}^{\alpha}\mathbf{e}^{\delta}. \end{aligned} \tag{6}$$

There are two another important geometric objects: the fractional Ricci tensor ${}^{\alpha}\mathcal{R}ic = \{ {}^{\alpha}\mathbf{R}_{\alpha\beta} \doteqdot {}^{\alpha}\mathbf{R}^{\tau}{}_{\alpha\beta\tau}\}$ with components

$$ {}^{\alpha}R_{ij} \doteqdot {}^{\alpha}R^{k}{}_{ijk}, \quad {}^{\alpha}R_{ia} \doteqdot -\ {}^{\alpha}R^{k}{}_{ika}, \quad {}^{\alpha}R_{ai} \doteqdot {}^{\alpha}R^{b}{}_{aib}, \quad {}^{\alpha}R_{ab} \doteqdot {}^{\alpha}R^{c}{}_{abc} \tag{7}$$

and the scalar curvature of fractional d–connection $\overset{\alpha}{\mathbf{D}}$,

$$ {}_{s}^{\alpha}\mathbf{R} \doteqdot {}^{\alpha}\mathbf{g}^{\tau\beta}\ {}^{\alpha}\mathbf{R}_{\tau\beta} = {}^{\alpha}R + {}^{\alpha}S,\ {}^{\alpha}R = {}^{\alpha}g^{ij}\ {}^{\alpha}R_{ij}, \quad {}^{\alpha}S = {}^{\alpha}g^{ab}\ {}^{\alpha}R_{ab}, \tag{8}$$

with ${}^{\alpha}\mathbf{g}^{\tau\beta}$ being the inverse coefficients to a d–metric (4).

We can introduce the Einstein tensor ${}^{\alpha}\mathcal{E}ns = \{ {}^{\alpha}\mathbf{G}_{\alpha\beta}\}$,

$$ {}^{\alpha}\mathbf{G}_{\alpha\beta} := {}^{\alpha}\mathbf{R}_{\alpha\beta} - \frac{1}{2}\,{}^{\alpha}\mathbf{g}_{\alpha\beta}\ {}^{\alpha}_{s}\mathbf{R}. \tag{9} $$

For various applications, we can consider more special classes of d–connections:

- There is a unique canonical metric compatible fractional d–connection ${}^{\alpha}\widehat{\mathbf{D}} = \{ {}^{\alpha}\widehat{\mathbf{\Gamma}}^{\gamma}_{\ \alpha\beta} = \left({}^{\alpha}\widehat{L}^{i}_{jk}, {}^{\alpha}\widehat{L}^{a}_{bk}, {}^{\alpha}\widehat{C}^{i}_{jc}, {}^{\alpha}\widehat{C}^{a}_{bc}\right)\}$, when ${}^{\alpha}\widehat{\mathbf{D}}\ ({}^{\alpha}\mathbf{g}) = 0$, satisfying the conditions ${}^{\alpha}\widehat{T}^{i}_{\ jk} = 0$ and ${}^{\alpha}\widehat{T}^{a}_{\ bc} = 0$, but ${}^{\alpha}\widehat{T}^{i}_{\ ja}$, ${}^{\alpha}\widehat{T}^{a}_{\ ji}$ and ${}^{\alpha}\widehat{T}^{a}_{\ bi}$ are not zero. The N–adapted coefficients are given in explicitly form in our works [1, 2, 3, 4, 5, 6].
- The fractional Levi–Civita connection ${}^{\alpha}\nabla = \{ {}^{\alpha}\Gamma^{\gamma}_{\ \alpha\beta}\}$ can be defined in standard from but for the fractional Caputo left derivatives acting on the coefficients of a fractional metric.

On spaces with nontrivial nonholonomic structure, it is preferred to work on $\overset{\alpha}{\mathbf{V}}$ with ${}^{\alpha}\widehat{\mathbf{D}} = \{ {}^{\alpha}\widehat{\mathbf{\Gamma}}^{\gamma}_{\ \tau\beta}\}$ instead of ${}^{\alpha}\nabla$ (the last one is not adapted to the N–connection splitting (1)). The torsion ${}^{\alpha}\widehat{\mathcal{T}}^{\tau}$ (6) of ${}^{\alpha}\widehat{\mathbf{D}}$ is uniquely induced nonholonomically by off–diagonal coefficients of the d–metric (4).

With respect to N–adapted fractional bases (2) and (3), the coefficients of the fractional Levi–Civita and canonical d–connection satisfy the distorting relations

$$ {}^{\alpha}\Gamma^{\gamma}_{\ \alpha\beta} = {}^{\alpha}\widehat{\mathbf{\Gamma}}^{\gamma}_{\ \alpha\beta} + {}^{\alpha}Z^{\gamma}_{\ \alpha\beta}, \tag{10} $$

where the N–adapted coefficients of distortion tensor $Z^{\gamma}_{\ \alpha\beta}$ are computed in Ref. [6].

An unified approach to Einstein–Lagrange/Finsler gravity for arbitrary integer and non–integer dimensions is possible for the fractional canonical d–connection ${}^{\alpha}\widehat{\mathbf{D}}$. The fractional gravitational field equations are formulated for the Einstein d–tensor (9), following the same principle of constructing the matter source ${}^{\alpha}\mathbf{\Upsilon}_{\beta\delta}$ as in general relativity but for fractional metrics and d–connections,

$$ {}^{\alpha}\widehat{\mathbf{E}}_{\ \beta\delta} = {}^{\alpha}\mathbf{\Upsilon}_{\beta\delta}. \tag{11} $$

Such a system of integro–differential equations for generalized connections can be restricted to fractional nonholonomic configurations for ${}^{\alpha}\nabla$ if we impose the additional constraints

$$ {}^{\alpha}\widehat{L}^{c}_{aj} = {}^{\alpha}e_{a}({}^{\alpha}N^{c}_{j}),\ \ {}^{\alpha}\widehat{C}^{i}_{jb} = 0,\ \ {}^{\alpha}\Omega^{a}_{\ ji} = 0. \tag{12} $$

There are not theoretical or experimental evidences that for fractional dimensions we must impose conditions of type (12) but they have certain physical motivation if we develop models which in integer limits result in the general relativity theory.

3 Exact Solutions in Fractional Gravity

We studied in detail [2] what type of conditions must satisfy the coefficients of a metric (4) for generating exact solutions of the fractional Einstein equations (11). For simplicity, we can use a "prime" dimension splitting of type $2+2$ when coordinated are labeled in the form $u^{\beta}=(x^j, y^3=v, y^4)$, for $i,j,...=1,2.$ and the metric ansatz has one Killing symmetry when the coefficients do not depend explicitly on variable y^4.

3.1 Separation of equations for fractional and integer dimensions

The solutions of equations can be constructed for a general source of type[3]

$$ {}^{\alpha}\Upsilon^{\alpha}{}_{\beta} = diag[\ {}^{\alpha}\Upsilon_{\gamma};\ {}^{\alpha}\Upsilon_1 = \ {}^{\alpha}\Upsilon_2 = \ {}^{\alpha}\Upsilon_2(x^k, v);\ {}^{\alpha}\Upsilon_3 = \ {}^{\alpha}\Upsilon_4 = \ {}^{\alpha}\Upsilon_4(x^k)] $$

For such sources and ansatz with Killing symmetries for metrics, the Einstein equations (11) can be integrated in general form.

We can construct 'non–Killing' solutions depending on all coordinates when

$$ \begin{aligned} {}^{\alpha}\mathbf{g} &= \ {}^{\alpha}g_i(x^k)\ {}^{\alpha}dx^i \otimes \ {}^{\alpha}dx^i + \ {}^{\alpha}\omega^2(x^j, v, y^4)\ {}^{\alpha}h_a(x^k, v)\ {}^{\alpha}\mathbf{e}^a \otimes \ {}^{\alpha}\mathbf{e}^a, \\ {}^{\alpha}\mathbf{e}^3 &= \ {}^{\alpha}dy^3 + \ {}^{\alpha}w_i(x^k, v)\ {}^{\alpha}dx^i,\ \ {}^{\alpha}\mathbf{e}^4 = \ {}^{\alpha}dy^4 + \ {}^{\alpha}n_i(x^k, v)\ {}^{\alpha}dx^i, \end{aligned} \quad (13) $$

for any ${}^{\alpha}\omega$ for which

$$ {}^{\alpha}\mathbf{e}_k\ {}^{\alpha}\omega = \overset{\alpha}{\underline{\partial}}_k\ {}^{\alpha}\omega + \ {}^{\alpha}w_k\ {}^{\alpha}\omega^* + \ {}^{\alpha}n_k \overset{\alpha}{\underline{\partial}}_{y^4}\ {}^{\alpha}\omega = 0. $$

Configurations with fractional Levi–Civita connection ${}^{\alpha}\nabla$, of type (12), can be extracted by imposing additional constraints

$$ \begin{aligned} {}^{\alpha}w_i^* &= \ {}^{\alpha}\mathbf{e}_i \ln|\ {}^{\alpha}h_4|,\ \ {}^{\alpha}\mathbf{e}_k\ {}^{\alpha}w_i = \ {}^{\alpha}\mathbf{e}_i\ {}^{\alpha}w_k, \\ {}^{\alpha}n_i^* &= 0,\ \overset{\alpha}{\underline{\partial}}_i\ {}^{\alpha}n_k = \overset{\alpha}{\underline{\partial}}_k\ {}^{\alpha}n_i, \end{aligned} \quad (14) $$

[3] such parametrizations of energy–momentum tensors are quite general ones for various types of matter sources

where the partial derivatives are

$$ {}^{\alpha}a^{\bullet} = \overset{\alpha}{\underline{\partial}}_1 a = {}_{1x^1}\overset{\alpha}{\underline{\partial}}_{x^1}{}^{\alpha}a, \quad {}^{\alpha}a' = \overset{\alpha}{\underline{\partial}}_2 a = {}_{1x^2}\overset{\alpha}{\underline{\partial}}_{x^2}{}^{\alpha}a, \quad {}^{\alpha}a^{*} = \overset{\alpha}{\underline{\partial}}_v a = {}_{1v}\overset{\alpha}{\underline{\partial}}_{v}{}^{\alpha}a, $$

being used the left Caputo fractional derivatives (A.3).

3.2 Solutions with ${}^{\alpha}h^{*}_{3,4} \neq 0$ and ${}^{\alpha}\Upsilon_{2,4} \neq 0$

For simplicity, we provide only a class of exact solution with metrics of type (13) when ${}^{\alpha}h^{*}_{3,4} \neq 0$ (in Ref. [2], there are analyzed all possibilities for coefficients[4]) We consider the ansatz

$$ \begin{aligned} {}^{\alpha}\mathbf{g} &= e^{\,{}^{\alpha}\psi(x^k)}\,{}^{\alpha}dx^i \otimes {}^{\alpha}dx^i + h_3(x^k,v)\,{}^{\alpha}\mathbf{e}^3 \otimes {}^{\alpha}\mathbf{e}^3 + h_4(x^k,v)\,{}^{\alpha}\mathbf{e}^4 \otimes {}^{\alpha}\mathbf{e}^4, \\ {}^{\alpha}\mathbf{e}^3 &= {}^{\alpha}dv + {}^{\alpha}w_i(x^k,v)\,{}^{\alpha}dx^i,\ {}^{\alpha}\mathbf{e}^4 = {}^{\alpha}dy^4 + {}^{\alpha}n_i(x^k,v)\,{}^{\alpha}dx^i \end{aligned} \qquad (15) $$

We consider any nonconstant ${}^{\alpha}\phi = {}^{\alpha}\phi(x^i,v)$ as a generating function. We have to solve respectively the two dimensional fractional Laplace equation, for ${}^{\alpha}g_1 = {}^{\alpha}g_2 = e^{\,{}^{\alpha}\psi(x^k)}$. Then we integrate on v, in order to determine ${}^{\alpha}h_3$, ${}^{\alpha}h_4$ and ${}^{\alpha}n_i$, and solve algebraic equations, for ${}^{\alpha}w_i$. We obtain (computing consequently for a chosen ${}^{\alpha}\phi(x^k,v)$)

$$ \begin{aligned} {}^{\alpha}g_1 &= {}^{\alpha}g_2 = e^{\,{}^{\alpha}\psi(x^k)},\ {}^{\alpha}h_3 = \pm \frac{|\,{}^{\alpha}\phi^{*}(x^i,v)|}{{}^{\alpha}\Upsilon_2}, \qquad (16) \\ {}^{\alpha}h_4 &= {}^{\alpha}_{0}h_4(x^k) \pm 2\ {}_{1v}\overset{\alpha}{I}_v \frac{(\exp[2\ {}^{\alpha}\phi(x^k,v)])^{*}}{{}^{\alpha}\Upsilon_2}, \\ {}^{\alpha}w_i &= -\overset{\alpha}{\underline{\partial}}_i\,{}^{\alpha}\phi/\,{}^{\alpha}\phi^{*},\ \ {}^{\alpha}n_i = {}^{\alpha}_{1}n_k\left(x^i\right) + {}^{\alpha}_{2}n_k\left(x^i\right)\ {}_{1v}\overset{\alpha}{I}_v[\,{}^{\alpha}h_3/(\sqrt{|\,{}^{\alpha}h_4|})^3], \end{aligned} $$

where ${}^{\alpha}_{0}h_4(x^k)$, ${}^{\alpha}_{1}n_k\left(x^i\right)$ and ${}^{\alpha}_{2}n_k\left(x^i\right)$ are integration functions, and ${}_{1v}\overset{\alpha}{I}_v$ is the fractional integral on variables v and

$$ \begin{aligned} {}^{\alpha}\phi &= \ln\left|\frac{{}^{\alpha}h_4^{*}}{\sqrt{|\,{}^{\alpha}h_3\,{}^{\alpha}h_4|}}\right|,\ \ {}^{\alpha}\gamma = \left(\ln|\,{}^{\alpha}h_4|^{3/2}/|\,{}^{\alpha}h_3|\right)^{*}, \qquad (17) \\ {}^{\alpha}\alpha_i &= {}^{\alpha}h_4^{*}\overset{\alpha}{\underline{\partial}}_k\,{}^{\alpha}\phi,\ \ {}^{\alpha}\beta = {}^{\alpha}h_4^{*}\ {}^{\alpha}\phi^{*}\ . \end{aligned} $$

For ${}^{\alpha}h_4^{*} \neq 0$; ${}^{\alpha}\Upsilon_2 \neq 0$, we have ${}^{\alpha}\phi^{*} \neq 0$. The exponent $e^{\,{}^{\alpha}\psi(x^k)}$ is the fractional analog of the "integer" exponential functions and called the

[4] by nonholonomic transforms, various classes of solutions can be transformed from one to another

Mittag–Leffer function $E_{\alpha}[(x - {}^{1}x)^{\alpha}]$. For ${}^{\alpha}\psi(x) = E_{\alpha}[(x - {}^{1}x)^{\alpha}]$, we have $\overset{\alpha}{\underline{\partial}}_i E_{\alpha} = E_{\alpha}$.

We have to constrain the coefficients (16) to satisfy the conditions (14) in order to construct exact solutions for the Levi–Civita connection ${}^{\alpha}\nabla$. To select such classes of solutions, we can fix a nonholonomic distribution when ${}_{2}^{\alpha}n_k\left(x^i\right) = 0$ and ${}_{1}^{\alpha}n_k\left(x^i\right)$ are any functions satisfying the conditions $\overset{\alpha}{\underline{\partial}}_i\ {}_{1}^{\alpha}n_k\left(xj\right) = \overset{\alpha}{\underline{\partial}}_k\ {}_{1}^{\alpha}n_i\left(x^j\right)$. The constraints on ${}^{\alpha}\phi(x^k, v)$ are related to the N–connection coefficients ${}^{\alpha}w_i = -\overset{\alpha}{\underline{\partial}}_i\ {}^{\alpha}\phi / \ {}^{\alpha}\phi^*$ following relations

$$
\begin{aligned}
(\ {}^{\alpha}w_i[\ {}^{\alpha}\phi])^* + \ {}^{\alpha}w_i[\ {}^{\alpha}\phi]\,(\ {}^{\alpha}h_4[\ {}^{\alpha}\phi])^* + \overset{\alpha}{\underline{\partial}}_i\ {}^{\alpha}h_4[\ {}^{\alpha}\phi] &= 0, \\
\overset{\alpha}{\underline{\partial}}_i\ {}^{\alpha}w_k[\ {}^{\alpha}\phi] = \overset{\alpha}{\underline{\partial}}_k\ \ {}^{\alpha}w_i[\ {}^{\alpha}\phi]&,
\end{aligned}
$$

where, for instance, we denoted by ${}^{\alpha}h_4[\ {}^{\alpha}\phi]$ the functional dependence on ${}^{\alpha}\phi$. Such conditions are always satisfied for ${}^{\alpha}\phi = \ {}^{\alpha}\phi(v)$ or if ${}^{\alpha}\phi = const$ when ${}^{\alpha}w_i(x^k, v)$ can be any functions with zero ${}^{\alpha}\beta$ and ${}^{\alpha}\alpha_i$, see (17)).

4 The Main Theorem on Fractional Solitonic Hierarchies

In Ref. [6, 7, 8], we proved that the geometric data for any fractional metric (in a model of fractional gravity or geometric mechanics) naturally define a N–adapted fractional bi–Hamiltonian flow hierarchy inducing anholonomic fractional solitonic configurations.

Theorem 4.1 *For any N–anholonomic fractional manifold with prescribed fractional d–metric structure, there is a hierarchy of bi-Hamiltonian N–adapted fractional flows of curves $\gamma(\tau, \mathbf{l}) = h\gamma(\tau, \mathbf{l}) + v\gamma(\tau, \mathbf{l})$ described by geometric nonholonomic fractional map equations. The 0 fractional flows are defined as convective (traveling wave) maps*

$$
\gamma_{\tau} = \gamma_{\mathbf{l}},\ \textit{distinguished}\ \ (h\gamma)_{\tau} = (h\gamma)_{h\mathbf{X}}\ \ \textit{and}\ \ (v\gamma)_{\tau} = (v\gamma)_{v\mathbf{X}}\,.
$$

There are fractional +1 flows defined as non–stretching mKdV maps

$$
\begin{aligned}
-(h\gamma)_{\tau} &= \ {}^{\alpha}\mathbf{D}^2_{h\mathbf{X}}\,(h\gamma)_{h\mathbf{X}} + \frac{3}{2}\,|\ {}^{\alpha}\mathbf{D}_{h\mathbf{X}}\,(h\gamma)_{h\mathbf{X}}|^2_{h\mathbf{g}}\ \ (h\gamma)_{h\mathbf{X}}\,, \\
-(v\gamma)_{\tau} &= \ {}^{\alpha}\mathbf{D}^2_{v\mathbf{X}}\,(v\gamma)_{v\mathbf{X}} + \frac{3}{2}\,|\ {}^{\alpha}\mathbf{D}_{v\mathbf{X}}\,(v\gamma)_{v\mathbf{X}}|^2_{v\mathbf{g}}\ \ (v\gamma)_{v\mathbf{X}}\,,
\end{aligned}
$$

and fractional +2,... flows as higher order analogs. Finally, the fractional -1 flows are defined by the kernels of recursion fractional operators inducing non–stretching fractional maps ${}^{\alpha}\mathbf{D}_{h\mathbf{Y}}\,(h\gamma)_{h\mathbf{X}} = 0$ *and* ${}^{\alpha}\mathbf{D}_{v\mathbf{Y}}\,(v\gamma)_{v\mathbf{X}} = 0.$

A Fractional Caputo N–anholonomic Manifolds

The fractional left, respectively, right Caputo derivatives are defined by formulas

$$
{}_{1x}\overset{\alpha}{\underline{\partial}}_{x} f(x) := \frac{1}{\Gamma(s-\alpha)} \int\limits_{1x}^{x} (x - x')^{s-\alpha-1} \left(\frac{\partial}{\partial x'}\right)^{s} f(x')dx'; \quad \text{(A.1)}
$$

$$
{}_{x}\overset{\alpha}{\underline{\partial}}_{\ 2x} f(x) := \frac{1}{\Gamma(s-\alpha)} \int\limits_{x}^{2x} (x' - x)^{s-\alpha-1} \left(-\frac{\partial}{\partial x'}\right)^{s} f(x')dx' \ .
$$

We can introduce $\overset{\alpha}{d} := (dx^j)^\alpha \quad {}_0\overset{\alpha}{\underline{\partial}}_j$ for the fractional absolute differential, where $\overset{\alpha}{d}x^j = (dx^j)^\alpha \frac{(x^j)^{1-\alpha}}{\Gamma(2-\alpha)}$ if ${}_1x^i = 0$. Such formulas allow us to elaborate the concept of fractional tangent bundle $\overset{\alpha}{\underline{T}}M,$ for $\alpha \in (0,1)$, associated to a manifold M of necessary smooth class and integer $\dim M = n$.[5]

Let us denote by $L_z({}_1x, {}_2x)$ the set of those Lesbegue measurable functions f on $[{}_1x, {}_2x]$ when $||f||_z = (\int\limits_{1x}^{2x} |f(x)|^z dx)^{1/z} < \infty$ and $C^z[{}_1x, {}_2x]$ be the space of functions which are z times continuously differentiable on this interval. For any real–valued function $f(x)$ defined on a closed interval $[{}_1x, {}_2x]$, there is a function $F(x) = {}_{1x}\overset{\alpha}{I}_x f(x)$ defined by the fractional Riemann–Liouville integral ${}_{1x}\overset{\alpha}{I}_x f(x) := \frac{1}{\Gamma(\alpha)} \int\limits_{1x}^{x} (x-x')^{\alpha-1} f(x')dx'$, when $f(x) = {}_{1x}\overset{\alpha}{\underline{\partial}}_x F(x)$, for all $x \in [{}_1x, {}_2x]$, satisfies the conditions

$$
\begin{aligned}
{}_{1x}\overset{\alpha}{\underline{\partial}}_x \left({}_{1x}\overset{\alpha}{I}_x f(x) \right) &= f(x),\ \alpha > 0, \qquad \text{(A.2)} \\
{}_{1x}\overset{\alpha}{I}_x \left({}_{1x}\overset{\alpha}{\underline{\partial}}_x F(x) \right) &= F(x) - F({}_1x),\ 0 < \alpha < 1.
\end{aligned}
$$

We can consider fractional (co) frame bases on $\overset{\alpha}{\underline{T}}M$. For instance, a fractional frame basis $\overset{\alpha}{\underline{e}}_\beta = e^{\beta'}_{\ \beta}(u^\beta)\overset{\alpha}{\underline{\partial}}_{\beta'}$ is connected via a vierlbein transform $e^{\beta'}_{\ \beta}(u^\beta)$ with a fractional local coordinate basis

$$
\overset{\alpha}{\underline{\partial}}_{\beta'} = \left(\overset{\alpha}{\underline{\partial}}_{j'} = {}_{1x^{j'}}\overset{\alpha}{\underline{\partial}}_{j'}, \overset{\alpha}{\underline{\partial}}_{b'} = {}_{1y^{b'}}\overset{\alpha}{\underline{\partial}}_{b'} \right), \qquad \text{(A.3)}
$$

[5] For simplicity, we may write both the integer and fractional local coordinates in the form $u^\beta = (x^j, y^a)$. We underlined the symbol T in order to emphasize that we shall associate the approach to a fractional Caputo derivative.

for $j' = 1, 2, ..., n$ and $b' = n+1, n+2, ..., n+n$. The fractional co–bases are related via $\underline{\overset{\alpha}{e}}^{\beta} = e^{\beta}_{\ \beta'}(u^{\beta})\overset{\alpha}{d}u^{\beta'}$, where

$$\overset{\alpha}{d}u^{\beta'} = \left((dx^{i'})^{\alpha}, (dy^{a'})^{\alpha}\right). \tag{A.4}$$

The fractional absolute differential $\overset{\alpha}{d}$ is written in the form

$$\overset{\alpha}{d} := (dx^j)^{\alpha} \quad {}_0\overset{\alpha}{\underline{\partial}}_j, \text{ where } \overset{\alpha}{d}x^j = (dx^j)^{\alpha}\frac{(x^j)^{1-\alpha}}{\Gamma(2-\alpha)},$$

where we consider ${}_1x^i = 0$.

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